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Stress-strain characteristics of particulate material under the condition of no lateral strain

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This paper deals with an application of generalized non-linear stress-strain relationships of an incremental nature for a particulate material in order to investigate the behaviour of a bed subjected to a loading cycle during which lateral strain is prevented. Initially, the limiting conditions in a bed of granular material are discussed, and a brief review presented of relevant work already carried out under the condition of no lateral strain. Both normally consolidated, and over-consolidated, beds are considered. The resulting calculations indicate that the stress history of the bed can have, depending on stress levels, a profound effect on behaviour during subsequent reloading. Both the state of stress and the stress–strain behaviour of the deposit are assessed. Finally, the results are shown to have an important application in a number of engineering problems, particularly where overconsolidated beds of material are under consideration.

List of symbols

The notation used in the theoretical work is as follows:

K_0 = coefficient of earth pressure at rest.

K_r = Rankine ratio $= (1 - \sin\phi)/(1 + \sin\phi)$

K_1, K_2, f_1, f_2, g = constants found experimentally

p_h = stress in any horizontal direction when $p_x = p_z = p_h$

$p_x \ldots$ = stress in co-ordinate directions

p_{xi}, p_{yi}, p_{zi} = stress levels in co-ordinate directions at commencement of unloading/reload-cycle

r_1, r_2, r_3 = stress ratios $= p_x$ (ultimate)$/p_{xi}$, p_y (ultimate)$/p_{yi}$, p_z (ultimate)$/p_{zi}$ respectively

$\delta p_y, \delta \epsilon_y$, etc = a small increment of the quantity concerned

$\epsilon_x \ldots$ = strains in co-ordinate directions

ϕ = angle of internal friction of soil

1. Introduction

The determination of the state of stress and strain in a particulate material is of considerable importance in a wide range of engineering applications. For example, the behaviour of a shallow bed, i.e. where the ratio of the depth to least lateral dimension is small, when successively loaded, unloaded and reloaded by a uniformly distributed surface pressure, is a basic problem which, although relatively simple, has not yet been satisfactorily solved. This example is important both in the packing of granular materials and a number of problems in the field of civil engineering. Shallow beds can occur as sedimentary soil deposits which are built up by an accumulation of sediments. This causes vertical compression of the soil at any given depth because of the increase in vertical stress. In many cases sedimentation occurs over a large area and the lateral strain can be taken as zero. Similar examples, although on a much smaller scale, can be seen in foundry work where the "squeezing" of shallow moulds is carried out by pressure compaction.

Before discussing the approaches to such problems in more detail, it may be instructive to discuss terminology often used in work of this type. One of the factors governing the ratio of lateral stress to vertical stress is the magnitude of the lateral strain. Depending on whether the bed has been "stretched" or "compressed" laterally, either naturally or artificially, this ratio can vary from the "active coefficient" to the "passive coefficient". These represent the limiting conditions that can take place in the granular bed. In the case of zero lateral strain the ratio of lateral to vertical stress is termed the "coefficient of earth pressure at rest", and the deposit is said to be in the "at rest" state, i.e.

$$p_h/p_y = K_0$$

where p_h = lateral stress; p_y = vertical stress; K_0 = coefficient of earth pressure at rest. Under the conditions outlined above it is clear that these stresses are principal stresses.

Jaky [1] has shown that the value of K_0 decreases with an increase in the angle of internal friction of the material, i.e.

$$K_0 = 1 - \sin \phi$$

where ϕ = angle of internal friction. This has generally been accepted as an upper bound solution for K_0 [2]. It should also be noted that it is satisfactory, as an upper bound solution, for the first loading of a particulate material, but can be seriously in error if applied to an overconsolidated bed.

A recent definition [3] specifies major and minor principal stresses and re-defines K_0 in terms of the ratio of the increment of the corresponding stresses, again with zero lateral strain. It is considered that this definition is only applicable under simplified stress conditions, and in cases of a bed subjected to a stress history will be incorrect. This point will be discussed in more detail at a later stage.

A widely used approach in the solution of problems involving a particulate media is the application of classical elasticity theory on the assumption that the material is homogeneous and isotropic. However, such assumptions are questionable and an elastic solution fulfilling a given set of boundary conditions can prove extremely cumbersome. In order to produce a fundamental approach to these problems the elastic and sliding strains must be simultaneously taken into account [4]. Using an analysis of this type it is possible to obtain the variation of K_0 when the particulate media is subjected to a varying stress field.

As discussed earlier, a sedimentary deposit is built up by a gradual accumulation of overburden. This corresponds to the first loading of the bed for which a value of K_0 can be determined using a variety of techniques. This value is generally less than 0.5. There is, however, evidence to suggest that the lateral stress can exceed the vertical stress if the bed is overconsolidated, i.e. heavily preloaded in the past. This has the tendency to produce lateral stresses which are substantially retained in the soil during the removal of the previously applied load. Under these circumstances the value of K_0 can be considerably in excess of unity, and it is the purpose of this investigation to determine the variation of K_0 in a bed with a given stress history.

This work is, therefore, of some importance for a number of applications which range from the prediction of soil pressure on retaining structures, particularly when heavy compaction of backfill is carried out, to the pressures used in the compaction and packing of granular media.

2. Theoretical investigations

Consider a semi-infinite bed of granular media with a horizontal free surface which has been normally consolidated during a sedimentation, or similar, process, such that the lateral strain is zero. Subsequently, the upper horizontal layers are gradually removed to produce an overconsolidated deposit, and this is followed by reloading. It is proposed to investigate the behaviour of a typical element of the bed during this process.

The equations used are those presented in a previous investigation [4].

2.1. Stage 1: first loading

The governing equations are [4]:

$$
\left.
\begin{aligned}
\delta\epsilon_x &= \frac{\delta p_x}{K_1\left(\dfrac{\sqrt{p_y p_z}}{K_r} - p_x\right)} - \frac{\delta p_y}{K_2 \dfrac{p_x}{p_z}\left(\dfrac{\sqrt{p_x p_z}}{K_r} - p_y\right)} - \frac{\delta p_z}{K_2 \dfrac{p_x}{p_y}\left(\dfrac{\sqrt{p_x p_y}}{K_r} - p_z\right)} \\
\delta\epsilon_y &= -\frac{\delta p_x}{K_2 \dfrac{p_y}{p_z}\left(\dfrac{\sqrt{p_y p_z}}{K_r} - p_x\right)} + \frac{\delta p_y}{K_1\left(\dfrac{\sqrt{p_x p_z}}{K_r} - p_y\right)} - \frac{\delta p_z}{K_2 \dfrac{p_y}{p_x}\left(\dfrac{\sqrt{p_x p_y}}{K_r} - p_z\right)} \\
\delta\epsilon_z &= -\frac{\delta p_x}{K_2 \dfrac{p_z}{p_y}\left(\dfrac{\sqrt{p_y p_z}}{K_r} - p_x\right)} - \frac{\delta p_y}{K_2 \dfrac{p_z}{p_x}\left(\dfrac{\sqrt{p_x p_z}}{K_r} - p_y\right)} + \frac{\delta p_z}{K_1\left(\dfrac{\sqrt{p_x p_y}}{K_r} - p_z\right)}
\end{aligned}
\right\}
\quad (1)
$$

for which $p_x \leqslant p_y \leqslant p_z$,

where p_x, ϵ_x = stress and strains respectively in co-ordinate directions; δp_x, $\delta\epsilon_x$, etc = a small increment of the quantity concerned; K_r = Rankine ratio; K_1, K_2 = constants found experimentally.

Examination of Equation 1 shows that because of the symmetry of the constituent equations the governing condition relating to the relative magnitude of the stress is not applicable when

$$\delta\epsilon_x = \delta\epsilon_z = 0; \text{ then } p_x = p_z = p_h$$

$$\delta p_y = \frac{\left(\dfrac{p_h}{K_r} - p_y\right)}{\left(\dfrac{\sqrt{p_y p_h}}{K_r} - p_h\right)} \left[\frac{K_2}{K_1} - \frac{p_y}{p_h}\right] \delta p_h.$$

For a typical sand, $K_1 = 45$; $K_2 = 1650$; and $K_r = 0.2$.

$$\delta p_y = \frac{(5p_h - p_y)}{(5\sqrt{(p_y p_h)} - p_h)} \left[36.7 - \frac{p_y}{p_h}\right] \delta p_h \quad (2)$$

It is important that the process of loading and the attainment of new stress states of the sample be examined. The point X in Fig. 1 represents the state of stress, i.e. a point on the $p_y - \epsilon_y$ curve, and the value of lateral stress is that necessary to prevent lateral strain. An incremental change δp_y in p_y causes a change in the state of stress and as this can only take place in the direction shown by the arrow the new stress state is X_1. However, to prevent lateral strain it is necessary to have a corresponding increment δp_h in lateral stress and this is equivalent to considering X_1 on a new p_h = constant curve in the $p_y - \epsilon_y$ stress space. This process is continued, to arrive at new states X_2 ... etc. until loading has been completed. It should be emphasized that the utilization of new $p_y - \epsilon_y$ loading curves represents a change in the orientation of the co-ordinate axes so that the discontinuous incremental stress path represented in Fig. 1 is transformed when plotted with reference to common axes.

Equation 2 represents the behaviour of the sample and the change of state is clearly dependent on the starting point in stress space. For first loading this has a trivial solution, i.e.

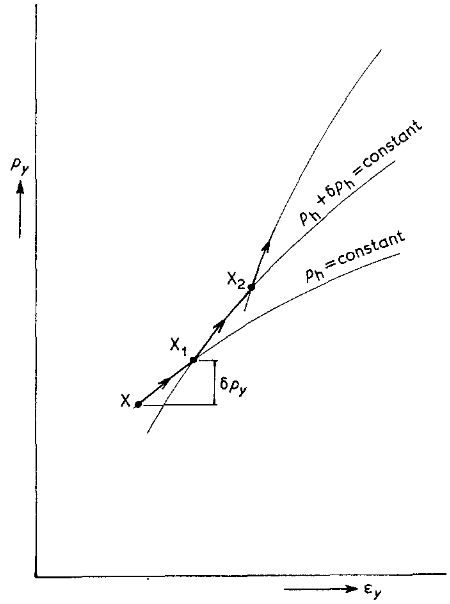

Figure 1 Incremental loading sequence.

$$\frac{1}{K_0}=\frac{(5K_0-1)}{(5\sqrt{K_0}-K_0)}\left(36.7-\frac{1}{K_0}\right),$$

hence $K_0 = 0.26$. This is consistent with Jaky's upper bound solution and, as would be expected, is greater than the active coefficient. In incremental form, for consistency with all other equations presented in this work:

$$\delta p_y = 3.85\,\delta p_h. \tag{3}$$

A point of interest is that some previous writers [3] have represented K_0 in this incremental form. This is only correct for the first loading case where

$$\frac{\delta p_h}{\delta p_y}=\frac{p_h}{p_y}=K_0$$

but in general, this is not applicable. Hence substituting back into Equation 1:

$$\delta\epsilon_y = \frac{1}{13.50\,p_y}\,\delta p_y. \tag{4}$$

This incremental equation represents the stress–strain relationship for the first loading.

2.2. Stage 2: unloading

Again from [4], the governing equations are:

$$\left.\begin{aligned}
\delta\epsilon_x &= \frac{\exp[-r_1p_x/\sqrt{(p_yp_z)}]}{f_1\sqrt{(p_yp_z)}}\,\delta p_x \\
&\quad-\frac{\exp[-r_2p_y/\sqrt{(p_xp_z)}]}{f_2(p_z/p_x)^g\sqrt{(p_xp_z)}}\,\delta p_y \\
&\quad-\frac{\exp[-r_3p_z/\sqrt{(p_xp_y)}]}{f_2(p_y/p_x)^g\sqrt{(p_xp_y)}}\,\delta p_z \\
\delta\epsilon_y &= -\frac{\exp[-r_1p_x/\sqrt{(p_yp_z)}]}{f_2(p_z/p_y)^g\sqrt{(p_yp_z)}}\,\delta p_x \\
&\quad+\frac{\exp[-r_2p_y/\sqrt{(p_xp_z)}]}{f_1\sqrt{(p_xp_z)}}\,\delta p_y \\
&\quad-\frac{\exp[-r_3p_z/\sqrt{(p_xp_y)}]}{f_2(p_x/p_y)^g\sqrt{(p_xp_y)}}\,\delta p_z \\
\delta\epsilon_z &= -\frac{\exp[-r_1p_x/\sqrt{(p_yp_z)}]}{f_2(p_y/p_z)^g\sqrt{(p_yp_z)}}\,\delta p_x \\
&\quad-\frac{\exp[-r_2p_y/\sqrt{(p_xp_z)}]}{f_2(p_x/p_z)^g\sqrt{(p_xp_z)}}\,\delta p_y \\
&\quad+\frac{\exp[-r_3p_z/\sqrt{(p_xp_y)}]}{f_1\sqrt{(p_xp_y)}}\,\delta p_z
\end{aligned}\right\} \tag{5}$$

for which $p_x \leqslant p_y \leqslant p_z$

where r_1, r_2 and r_3 are stress ratios defined by p_x (ultimate)/p_{xi}, p_y (ultimate)/p_{yi} and p_z (ultimate/p_{zi} respectively, p_{xi}, p_{yi} and p_{zi} are the stress levels at which unloading commences, and f_1, f_2, g are constants found experimentally.

As it is required to find K_0 when unloading, then

$$\delta\epsilon_x=\delta\epsilon_z=0;\text{ and } p_x=p_z=p_h.$$

$$\therefore\quad \frac{\exp[-r_2p_y/p_h]}{f_2p_h}\,\delta p_y=\left[\frac{\exp[-r_1\sqrt{(p_h/p_y)}]}{f_1\sqrt{(p_yp_h)}}-\frac{\exp[-r_3\sqrt{(p_h/p_y)}]}{f_2(p_y/p_h)^g\sqrt{(p_yp_h)}}\right]\delta p_h.$$

As p_x (ultimate) $= p_z$ (ultimate) $= p_h$ (ultimate) $= \sqrt{(p_yp_h)}/K_r$
and p_y (ultimate) $= \sqrt{(p_xp_z)}/K_r = p_h/K_r$

$$\therefore\quad r_1=r_3=\frac{\sqrt{(p_yp_h)}}{K_rp_{hi}}$$

and

$$r_2=\frac{p_h}{K_rp_{yi}}.$$

At the commencement of unloading, $p_y = p_{yi}$ and $ph = p_{hi}$; hence the above equation reduces to:

$$\frac{1}{f_2p_h}\,\delta p_y=\frac{1}{\sqrt{(p_yp_h)}}\left[\frac{1}{f_1}-\frac{1}{f_2(p_y/p_h)^g}\right]\delta p_h;$$

for a typical sand, $f_1 = 70$; $f_2 = 1470$; $g = 0$.

By substitution into the above equation it follows that:

$$\delta p_y = 20\sqrt{\left(\frac{p_h}{p_y}\right)}\,\delta p_h \tag{6}$$

and this is valid for $p_y/p_h \geqslant 0.2$.

Now it is clear that Equation 6 applies for the first increment of unloading from the first loading state. Referring to Fig. 2, it is instructive to examine the behaviour of the element of material during this first increment of unloading.

The state of stress of the material immediately before unloading can be represented by Y, but it must be emphasized that the approach to this state was made on a number of p_h = constant loading paths separated by the chosen increment of lateral stress δp_h.

A change in the vertical stress δp_y causes a movement in stress space from Y → Y$_1$ but in order to prevent lateral strain, a corresponding increment in lateral stress δp_h must take place. This is equivalent to considering Y$_1$ on a new p_h = constant

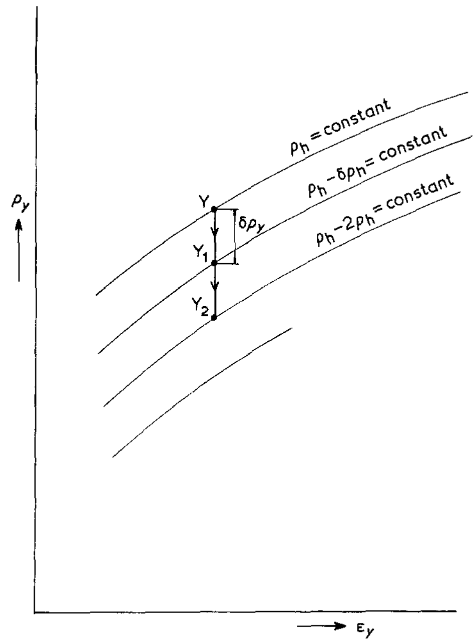

Figure 2 Incremental unloading sequence.

curve and the next increment of unloading can be referred to this as an initial state. It is, therefore, evident that Equation 6 applies continually during the unloading process.

Substituting Equation 6 into Equation 5:

$$\delta\epsilon_y = \frac{0.0142}{p_h} e^{-1/K_r}\, \delta p_y. \qquad (7)$$

This incremental equation represents the stress–strain relationship for the unloading.

2.3. Stage 3: reloading

The same process is followed as described for the first loading sequence except that as the starting point is any general point in stress space, the solution is non-trivial. Therefore, Equation 2 is applicable to the reloading, but in general,

$$\frac{\delta p_h}{\delta p_y} \neq \frac{p_h}{p_y},$$

although it will be seen later that at stress levels that are correspondingly high compared with those "locked in" during the unloading process there will be, as anticipated, a tendency to re-establish this condition.

Substituting Equation 2 into Equation 1 and simplifying:

$$\delta\epsilon_y = \frac{1}{45(5p_h - p_y)} \left[1 - \frac{p_h}{18.35 p_y \left(36.7 - \dfrac{p_y}{p_h}\right)}\right] \delta p_y. \qquad (8)$$

This incremental equation represents the stress–strain relationship for the reloading.

This will be found to approximate to

$$\delta\epsilon_y = \frac{1}{45(5p_h - p_y)}\, \delta p_y$$

and also as $p_y \to \infty$, $K_0 \to 0.26$; then

$$\delta\epsilon_y = \frac{1}{13.50 p_y}\, \delta p_y$$

which is identical to Equation 4 for the first loading.

Summarizing the previous work it can be seen that the variation of K_0 during the loading/unloading and reloading cycle is represented by Equations 3, 6 and 2 respectively. The stress–strain relationships applicable under these conditions are represented by Equations 4, 7 and 8. Because the lateral strain is equal to zero the volume changes associated

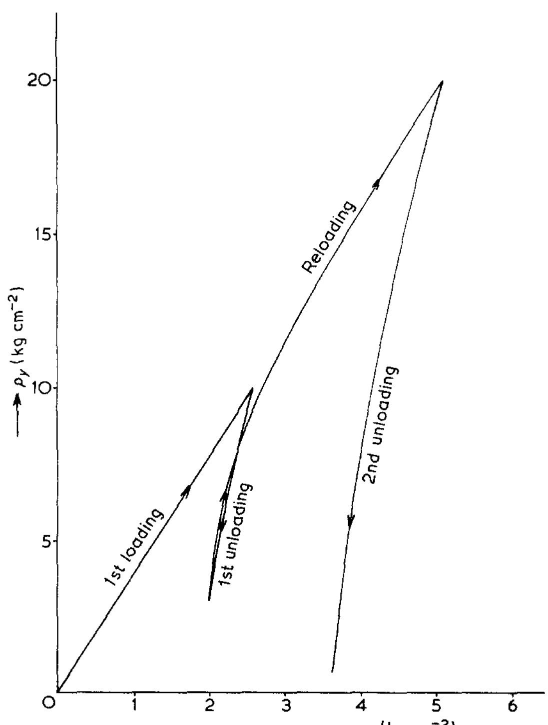

Figure 3 Relationship between vertical and lateral stresses.

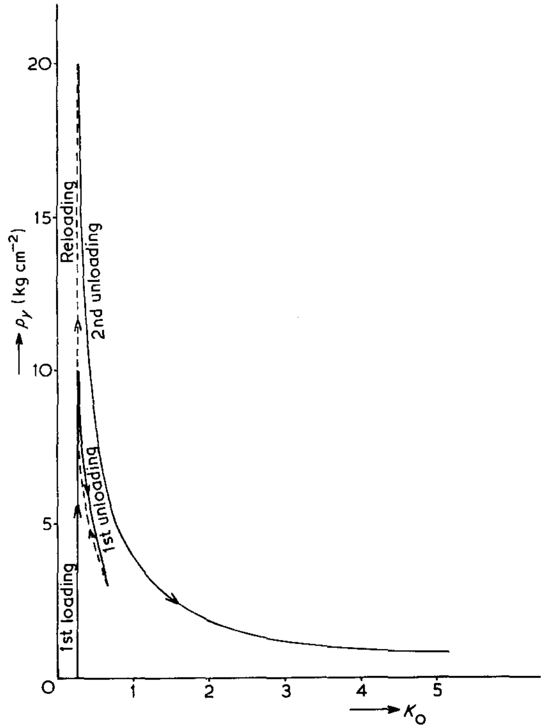

Figure 4 Variation of K_0 during loading cycle.

with the loading sequence are proportional to the strain ϵ_y.

Using these equations, the behaviour of a sample has been investigated under a typical loading cycle of the following description.

First loading: $p_y = 0$ to $p_y = 10$ kg cm^{-2}
First unloading: $p_y = 10$ kg cm^{-2} to $p_y = 3$ kg cm^{-2}
Reloading: $p_y = 3$ kg cm^{-2} to $p_y = 20$ kg cm^{-2}
Second unloading: $p_y = 20$ kg cm^{-2} to $p_y = 0.7$ kg cm^{-2}
at this stage $p_h/p_y = 1/K_r = 5.0$; which is the limiting condition for the material.

The variation of p_h with p_y is shown in Fig. 3. Fig. 4 shows the variation of K_0 associated with the changing stress field. Finally, Fig. 5 represents the stress–strain behaviour of the sample, which also is effectively a plot of volume changes. It should be noted that as this is applicable to a typical element, in order to find the bed behaviour it would be necessary to investigate the volumetric changes at all levels in the bed based on their respective stress histories. The stress–strain equations appeared to be the most sensitive to variation of the magnitude of incremental changes in the numerical work, particularly during the initial stages of the incremental process.

Discussion of results

During the first loading, the graph of p_h against p_y is a straight line thus yielding a constant value for K_0. Of interest is that the stress–strain relationship shows an increase in stiffness of the material when applied stresses are increased. This is also an indication of the relative behaviour of samples at different depths, i.e. the stiffness increases with the depth of the bed.

When the sample is subjected to unloading the vertical stress decreases rapidly with relatively little corresponding decrease in the lateral stress. Thus the vertical stress is considerably relieved leaving substantial lateral stress "locked in" the material. This results in a corresponding rise in K_0. The stress–strain relationship in this region is virtually linear and shows very little strain relief, which is consistent with the substantial lateral stresses that have been retained. When the relief in vertical stress is sufficient to produce a value of $K_0 > 5.0$, as in the second unloading, the equation becomes invalid as this represents the limiting condition in the material. It suggests that there is a thin layer on the surface of the granular mass which undergoes shear failure and consequent stress relaxation when the applied load is removed. However,

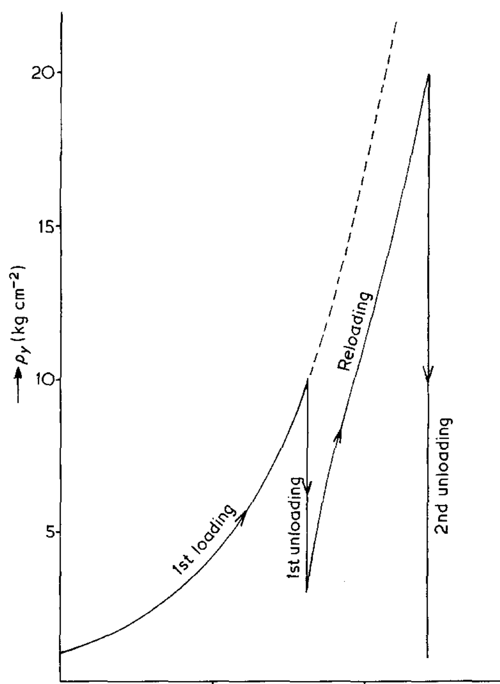

Figure 5 Stress–strain behaviour during loading cycle.

at points not immediately adjacent to the free surface the vertical stress is maintained, and hence the predicted stress–strain behaviour applicable.

Finally, the anticipated factor in the reloading of the material is that as the vertical stress is substantially increased, causing a corresponding increase in the lateral stress, the value of K_0 and the stress–strain behaviour approach that predicted by the first loading. This indicates that under certain circumstances, at least for a material in the "at rest" state, the stress history of the material can be sufficiently overwritten to be of minor importance.

4. Conclusions

Much of the work done in this area of study has dealt with the experimental measurement of K_0 under first loading conditions, and the effect of the different constituents in the samples tested [5]. Elastic approaches have also been used to attempt to predict K_0 under simplified conditions. The importance of the present work is that it uses a fundamental expression taking into account both the elastic and sliding strains to determine K_0, not only for first loading, but throughout the complete range of loading.

The advantages of such an approach are of some importance. These include a more realistic assessment of the earth pressures on retaining structures. It has been acknowledged [6] that if a cantilever wall has a foundation which allows little or no movement or rotation, the active conditions may not develop without causing sever cracking of the concrete and yielding of the reinforcing steel. Cantilever retaining walls should, therefore, be designed on the basis of the "at rest" rather than active condition. There are other examples of unyielding walls and the situation is more complex when compaction has been carried out adjacent to the structure. With heavy compaction, conditions approaching the passive may well be developed. The calculations presented in this study would determine the design conditions for the structure.

Another application could be a more accurate assessment of the shaft resistance when assessing pile and anchor capacities. Equations similar to the type presented have already been used in relevant cavitation calculations [7] which are directly applicable to this area of study. The added advantages of more closely predicting the behaviour of these systems in overconsolidated soil masses are of considerable interest, as the assessment of the lateral stresses which contribute to the determination of shaft friction has been a matter of some speculation.

Finally, there may be an application in the shallow bed pressing and packing [8] of particulate materials for a range of industrial products.

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